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EVERY SEMIGROUP IS ISOMORPHIC TO A TRANSITIVE SEMIGROUP OF BINARY RELATIONS

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ABSTRACT. Every (finite) semigroup is isomorphic to a transitive semigroup of binary relations (on a finite set).

Let $\mathcal{B}_A$ be the set of all binary relations between elements of a set A. We consider $\mathcal{B}_A$ as a semigroup with the operation of relative product $\circ$. Its subsemigroups are called *semigroups of binary relations.* A *(faithful) representation* of a semigroup $\mathcal{S}$ by relations is a(n injective) homomorphism of $\mathcal{S}$ into $\mathcal{B}_A$, A being any set.

A subset $\Phi \subset \mathcal{B}_A$ is called *transitive* if $\bigcup \Phi = A \times A$ (that is, for any $a, b \in A$ there exists $\varphi \in \Phi$ with $(a, b) \in \varphi$). A representation P of a semigroup $\mathcal{S}$ is called *transitive* if $\mathrm{P}(S)$ is a transitive set of relations, that is, P can be viewed as a homomorphism of $\mathcal{S}$ onto a transitive semigroup of relations. A longstanding problem of semigroup theory (see [4]) asks which semigroups have faithful transitive representations by relations. An equally longstanding conjecture is: *all.* Various classes of semigroups (subdirectly irreducible, with zero, completely [0]-simple) were proved to have faithful transitive representations by relations (see [4] and [5]). The main results of this paper are the following theorems.

Theorem A. *Every semigroup is isomorphic to a transitive semigroup of binary relations.*

Theorem B. *Every finite semigroup is isomorphic to a transitive semigroup of binary relations on a finite set.*

Before proving the theorems, we mention some open problems.

Open Problems. 1. A relation $\varphi \subset A \times A$ is called a *multipermutation* if its domain and range coincide with A, that is, if, given any $a \in A$, there exist $b, c \in A$ with $(a, b), (c, a) \in \varphi$. Every semigroup is isomorphic to a semigroup of multipermutations [4]. *Which semigroups are isomorphic to transitive semigroups of multipermutations?*

2. Every set Φ of binary relations is ordered by the inclusion relation $\subset$, and every semigroup $(\Phi; \circ)$ of relations becomes an ordered semigroup $(\Phi; \circ; \subset)$. Speaking of orders, we always mean *partial* orders. Clearly, $\subset$ is a *stable* order on Φ (that is, $\subset$ is a subsemigroup of the semigroup $\Phi \times \Phi$ or, equivalently,

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$\varphi_1 \subset \psi_1 \wedge \varphi_2 \subset \psi_2 \Rightarrow \varphi_1 \circ \psi_1 \subset \varphi_2 \circ \psi_2$ for all $\varphi_i, \psi_j \in \Phi$. Here and elsewhere $\wedge$ is a symbol of logical conjunction "and"). Every ordered semigroup $(S; \cdot; \leq)$ (by definition, $\leq$ is stable) is isomorphic to an ordered semigroup $(\Phi; \circ; \subset)$ of relations [9]; that is, there exists a bijection $R : S \to \Phi$ with $R(st) = R(s) \circ R(t)$ and $s \leq t \Leftrightarrow R(s) \subset R(t)$ for all $s, t \in S$. We can choose R preserving all existing infima of nonempty subsets of S (turning them into intersections of the corresponding binary relations) [1]. *Which ordered semigroups are isomorphic to transitive semigroups of binary relations ordered by set-theoretical inclusion?*

Example. Let $S = \{0, 1\}$ be a two-element zero semigroup ($xy = 0$ for all $x, y \in S$) with the order $0 < 1$. Then $\mathcal{S} = (S; \cdot; \leq)$ is an ordered semigroup. If R is an isomorphism of $\mathcal{S}$ onto a transitive inclusion-ordered semigroup of relations on a set A, then, since R is transitive and $R(0) \subset R(1)$, we obtain $R(1) = R(0) \cup R(1) = A \times A$. It follows that

$$R(0) = R(1^2) = R(1) \circ R(1) = (A \times A) \circ (A \times A) = A \times A = R(1)\,,$$

contradicting the faithfulness of R. Therefore, $\mathcal{S}$ is an ordered semigroup that is not isomorphic to any transitive inclusion-ordered semigroup of relations.

3. Every ordered semigroup is isomorphic to a semigroup of multipermutations ordered by inclusion [4]. *Which ordered semigroups are isomorphic to transitive semigroups of multipermutations ordered by inclusion?*

The remaining part of this paper is devoted to the proofs. The proof of Theorem A is based on constructions that may be of independent interest. This approach is discussed briefly after Theorem A is proved. For finite semigroups our constructions may yield a representation by relations on an infinite set. This is why Theorem B is given a separate proof based on a somewhat different idea.

Proof of Theorem A

Let A be a set, $\mathcal{S} = (S; \cdot)$ a semigroup, and $\mu : S \to \mathcal{B}_A$ a mapping. We call μ *transitive* if $\mu(S)$ is a transitive subset of $\mathcal{B}_A$. We call μ a *quasi-representation* if $\mu(s) \circ \mu(t) \subset \mu(st)$ for all $s, t \in S$. Thus a *representation* is a quasi-representation R such that $R(s) \circ R(t) \supset R(st)$. We will extend μ to a representation R_ω of $\mathcal{S}$ by relations on a set $A_\omega \supset A$. This is done in two steps.

Step I. Here we define a quasi-representation Q_μ. It is helpful to use the graph approach introduced now. Let S and V be sets called the sets of *labels* and *vertices*, respectively. A *labeled multi-graph* is any mapping $\mu : S \to \mathcal{B}_V$. A *labeled arrow* is a triple (i, s, j) such that $(i, j) \in \mu(s)$. It is interpreted as an (oriented) arrow from i to j labeled by s, graphically $i \xrightarrow{s} j$. Alternatively, a labeled multigraph can be defined as just a subset of $V \times S \times V$.

A *labeled path* from i to j is a sequence of labeled arrows

$$\pi = \big((i_0, s_0, i_1), (i_1, s_1, i_2), ..., (i_{n-1}, s_{n-1}, i_n)\big),$$

where $i = i_0$ and $j = i_n$. Graphically, π is

$$i = i_0 \xrightarrow{\ s_0\ } i_1 \xrightarrow{\ s_1\ } i_2 \xrightarrow{\ s_2\ } \cdots \xrightarrow{\ s_{n-2}\ } i_{n-1} \xrightarrow{\ s_{n-1}\ } i_n = j.$$

To define the length of π we combine the labels $s_0, ..., s_{n-1}$ in their consecutive order assuming that the set S of labels is endowed with a binary operation $\cdot$, that is, $\mathcal{S} = (S; \cdot)$ is a groupoid. The product $l(\pi) = s_0 \cdot s_1 \cdot ... \cdot s_{n-1}$ is called the *length* of the path π. In particular, the label s of a labeled arrow (i, s, j) is its length. We assume that $\cdot$ is associative, that is, $\mathcal{S}$ is a semigroup, so that $l(\pi)$ does not depend on the way in which parentheses are placed in $s_0 \cdot s_1 \cdot \ldots \cdot s_{n-1}$.

If $\pi_1 : i = i_0 \xrightarrow{*} \cdots \xrightarrow{*} i_m = j$ and $\pi_2 : j = j_0 \xrightarrow{*} \cdots \xrightarrow{*} j_n = k$ are labeled paths from $i = i_0$ to $j = i_m$ and from $j = j_0$ to $k = j_n$, respectively, their *concatenation* is the following labeled path $\pi_1\pi_2$ from i to k :

$$\pi_1\pi_2 : i = i_0 \xrightarrow{*} i_1 \cdots \xrightarrow{*} i_m = j_0 \xrightarrow{*} \cdots \xrightarrow{*} j_n = k\,.$$

Obviously, $l(\pi_1\pi_2) = l(\pi_1)l(\pi_2)$.

Remark on notation. A relative product $\psi \circ \varphi$ can be defined either as $(a,c) \in \psi \circ \varphi \Leftrightarrow (a,b) \in \varphi \wedge (b,c) \in \psi$ for some b, or as $(a,c) \in \psi \circ \varphi \Leftrightarrow (a,b) \in \psi \wedge (b,c) \in \varphi$ for some b. If φ and ψ are mappings and $\varphi(a) = b$ stands for $(a,b) \in \varphi$, then $\psi \circ \varphi$ is a mapping such that $(\psi \circ \varphi)(a) = \psi(\varphi(a))$, and hence one should adhere to the former definition of $\circ$. This is why one of us used the former definition of $\circ$ in his previous publications. If we adhere to the former notation with the factors in a relative product written from the right to the left, and read the factors st in a product of elements of an abstract semigroup from the left to the right, we have to define a representation by the equality $\mathrm{P}(t) \circ \mathrm{P}(s) = \mathrm{P}(st)$. This seeming contradiction to our definition can be avoided if $\varphi(a)$ is replaced by $a\varphi$ or $(a)\varphi$, in which case $(a)(\varphi \circ \psi) = ((a)\varphi)\psi$. Another possibility is interpreting $\varphi(a) = b$ as $(b,a) \in \varphi$. However, this does not lie squarely with the habitual graph notation $a \xrightarrow{\varphi} b$ for $\varphi(a) = b$. Thus, we may want to consider t as the first factor in the product st.

Another possibility is using the latter definition of $\circ$, writing $\psi\varphi$ instead of $\psi \circ \varphi$. This notation is used in the theory of relations, but it is awkward in the semigroup context. Indeed, if T and U are subsets of a semigroup $\mathcal{S}$, then TU ordinarily stands for the subset $\{tu \mid t \in T, u \in U\}$. If φ and ψ are relations between elements of a semigroup $\mathcal{S}$, then they are subsets of $S \times S$, which is a semigroup (the direct product of two copies of $\mathcal{S}$). Thus, if we want to adhere to conventional notation, we have to conclude that $\varphi\psi = \{(su, tv) \mid (s,t) \in \varphi \wedge (u,v) \in \psi\}$. For example, if φ is an order relation on a semigroup $\mathcal{S}$, then $\varphi\varphi \subset \varphi$ means that φ is stable (that is, "compatible with multiplication," $s \le t \wedge u \le v \Rightarrow su \le tv$, where $a \le b$ stands for $(a,b) \in \varphi$). On the other hand, $\varphi \circ \varphi \subset \varphi$ means that φ is transitive (that is, $(a,b) \in \varphi \wedge (b,c) \in \varphi \Rightarrow (a,c) \in \varphi$).

In this paper we assume that $(a,c) \in \psi \circ \varphi \Leftrightarrow (a,b) \in \psi \wedge (b,c) \in \varphi$ for some b. Yet, contrary to what we have just said, we write $\mathrm{P}(s)$ instead of more logical $(s)\mathrm{P}$. Mathematics is both logical and consequential, but who said that mathematicians should be?

Let $\mathcal{S}$ be a semigroup and $\mu : S \to \mathcal{B}_V$ a mapping. Define a new mapping $Q_\mu : S \to \mathcal{B}_V$ as follows:

$$Q_\mu(s) = \bigcup \{\mu(s_1) \circ \mu(s_2) \circ \cdots \circ \mu(s_n) \mid n \ge 1,\ s = s_1 s_2 \ldots s_n\}.$$

Thus $(i,j) \in Q_\mu(s)$ when there exists a path of length s leading from i to j.

Lemma 1. *Q_μ is a quasi-representation. If μ is transitive, so is Q_μ.*

Proof. Let $(i,k) \in Q_\mu(s) \circ Q_\mu(t)$. Then there exists j such $(i,j) \in Q_\mu(s)$ and $(j,k) \in Q_\mu(t)$, that is, there exist paths π_1 from i to j and π_2 from j to k such that $l(\pi_1) = s$ and $l(\pi_2) = t$. The concatenation $\pi_1\pi_2$ is a path from i to k and $l(\pi_1\pi_2) = l(\pi_1)l(\pi_2) = st$, so that $(i,k) \in Q_\mu(st)$. It follows that $Q_\mu(s) \circ Q_\mu(t) \subset$

$Q_\mu(st)$, and hence Q_μ is a quasi-representation of $\mathcal{S}$. Obviously, if μ is transitive, then Q_μ is transitive. □

Step II. Now we extend quasi-representations to representations. Let $\mu : S \to \mathcal{B}_V$ be a quasi-representation of a semigroup $\mathcal{S}$. For every $i, j \in V$ and $s, t \in S$ such that $(i,j) \in \mu(st)$ but $(i,j) \notin \mu(s) \circ \mu(t)$, add a new vertex $k_{i,j,s,t}$ to V. This extends V to a larger set $\bar{V}$ of vertices. For each new vertex $k_{i,j,s,t}$ add two new labeled arrows $(i, s, k_{i,j,s,t})$ and $(k_{i,j,s,t}, t, j)$ (graphically $i \xrightarrow{s} k_{i,j,s,t} \xrightarrow{t} j$,) to the multi-graph μ, thus obtaining a new multi-graph $\bar{\mu}$. Finally, extend $\bar{\mu}$ to a quasi-representation $Q_{\bar{\mu}}$ as described in Step I.

Definition. Let $Q : S \to \mathcal{B}_A$ and $R : S \to \mathcal{B}_B$ be quasi-representations of a semigroup $\mathcal{S}$, $A \subset B$ and $Q(s) = R(s) \cap (A \times A)$ for all $s \in S$. Then R is called an *extension* of Q.

Lemma 2. *$Q_{\bar{\mu}}$ is an extension of μ. If μ is transitive, so is $Q_{\bar{\mu}}$.*

Proof. The inclusion $\mu(r) \subset Q_{\bar{\mu}}(r) \cap (A \times A)$ is obvious. If $(a,b) \in Q_{\bar{\mu}}(r) \cap (A \times A)$, there exists a path π from a to b of length r. The endpoints a and b of π belong to the old set of vertices A. Therefore, new vertices can appear only in subpaths $i \xrightarrow{s} k_{i,j,s,t} \xrightarrow{t} j$ of π. Replacing each of these subpaths by a labeled arrow $i \xrightarrow{st} j$, which belongs to the old multi-graph μ, we obtain a new path π' of the same length as π and with the same endpoints but without new vertices, so that π' is a path in the old multi-graph μ. It follows that $(a,b) \in \mu(r)$, and hence $\mu(r) = Q_{\bar{\mu}}(r) \cap (A \times A)$.

Suppose that μ is transitive and $a, b \in \bar{A}$. If $a, b \in A$, then $(a,b) \in \mu(u)$ for some $u \in S$, and hence $(a,b) \in Q_{\bar{\mu}}(u)$. If $a \in A$ and $b \notin A$, then $b = k_{i,j,s,t}$ for some $i, j \in A$ and $s, t \in S$. Since μ is transitive, $(a,i) \in \mu(v)$ for some $v \in S$. It follows that $a \xrightarrow{v} i \xrightarrow{s} b$ is a path in $\bar{\mu}$, and hence $(a,b) \in Q_{\bar{\mu}}(vs)$. Analogously, if $a \notin A$ and $b \in A$, then $a = k_{i,j,s,t}$, also $(j,b) \in \mu(v)$ for some $v \in S$, and hence $(a,b) \in Q_{\bar{\mu}}(t) \circ Q_{\bar{\mu}}(v) \subset Q_{\bar{\mu}}(tv)$. Finally, if $a, b \notin A$, then $a = k_{i_1,j_1,s_1,t_1}$ and $b = k_{i_2,j_2,s_2,t_2}$ for some $i_1, i_2, j_1, j_2 \in A$ and $s_1, s_2, t_1, t_2 \in S$. Since μ is transitive, $(j_1, i_2) \in \mu(u)$ for some $u \in S$. Thus $a \xrightarrow{t_1} j_1 \xrightarrow{u} i_2 \xrightarrow{s_2} b$ is a path in $\bar{\mu}$, and hence $(a,b) \in Q_{\bar{\mu}}(t_1 u s_2)$. Therefore, $Q_{\bar{\mu}}$ is transitive. □

If $R : S \to \mathcal{B}_A$ is a quasi-representation of $\mathcal{S}$, define inductively A_n and $R_n : S \to \mathcal{B}_{A_n}$ as follows. Let $A_0 = A$ and $R_0 = R$. If A_n and $R_n : S \to \mathcal{B}_{A_n}$ have already been defined, let $A_{n+1} = \bar{A}_n$ and $R_{n+1} = Q_{\bar{R}_n}$. By Lemma 2, R_n is a quasi-representation of $\mathcal{S}$ by binary relations on A_n for every $n \geq 0$.

Now define A_ω and R_ω as follows:

$$A_\omega = \bigcup_{n=0}^{\infty} A_n \text{ and } R_\omega(s) = \bigcup_{n=0}^{\infty} R_n(s) \text{ for all } s \in S\,.$$

Lemma 3. *R_ω is a representation of $\mathcal{S}$ by binary relations on A_ω and an extension of R_0. If R_0 is transitive, so is R_ω.*

Proof. If $(i,j) \in R_\omega(s) \circ R_\omega(t)$ for some $i, j \in A_\omega$ and $s, t \in S$, then $(i,k) \in R_\omega(s)$ and $(k,j) \in R_\omega(t)$ for some $k \in A_\omega$. It follows from the definition of R_ω that $(i,k) \in R_m(s)$ and $(k,j) \in R_n(t)$ for some $m, n \geq 0$. If $p \geq m$ and $p \geq n$, then $R_m(s) \subset R_p(s)$ and $R_n(t) \subset R_p(t)$, so that $(i,k) \in R_p(s)$ and $(k,j) \in R_p(t)$, whence $(i,j) \in R_p(s) \circ R_p(t) \subset R_p(st) \subset R_\omega(st)$, because R_p is a quasi-representation. Thus R_ω is a quasi-representation.

Now let $(i,j) \in R_\omega(st)$. Then $(i,j) \in R_n(st)$ for some $n \geq 0$. There may exist $k \in A_n$ such that $(i,k) \in R_n(s)$ and $(k,j) \in R_n(t)$. If no such k exists in A_n, then there exists $k = k_{i,j,s,t} \in A_{n+1}$ such that $(i,k) \in R_{n+1}(s)$ and $(k,j) \in R_{n+1}(t)$. Since R_ω is an extension of both R_n and R_{n+1}, we obtain $(i,k) \in R_\omega(s)$ and $(k,j) \in R_\omega(t)$, and hence $(i,j) \in R_\omega(s) \circ R_\omega(t)$. Therefore, $R(st) \subset R_\omega(s) \circ R_\omega(t)$. It follows that R_ω is a representation of $\mathcal{S}$. Also,

$$R_\omega(s) \cap A_m \times A_m = \bigcup_{n=0}^{\infty} \big(R_n(s) \cap A_m \times A_m\big) = R_m(s),$$

so that R_ω is an extension of R_m for all $m \geq 0$.

If R_0 is transitive, then R_n is transitive for all $n \geq 0$ by induction on n and Lemma 2. If $i, j \in A_\omega$, then $i \in A_m$ and $j \in A_n$ for some m and n, and hence $i, j \in A_p$ for every p such that $p \geq m$ and $p \geq n$. Since R_p is transitive, there exists $s \in S$ such that $(i,j) \in R_p(s) \subset R_\omega(s)$. It follows that R_ω is transitive. $\square$

Definition. Extend a labeled multi-graph $\mu : S \to \mathcal{B}_A$ to a quasi-representation $R = Q_\mu$ using Step I. Using Step II, extend R to a representation R_ω of $\mathcal{S}$. We call R_ω a *free representation of $\mathcal{S}$ generated by μ.*

Suppose that Φ is a transitive semigroup of relations on a set A and Ψ an ideal of Φ that contains a nonempty relation ψ (this is so if, for example, Ψ is a nonzero ideal). Let $(a,b) \in \psi$. If $i, j \in A$, then $(i,a) \in \alpha$ and $(b,j) \in \beta$ for some $\alpha, \beta \in \Phi$, because Φ is transitive. It follows that $(i,j) \in \alpha \circ \psi \circ \beta \in \Psi$, and hence Ψ is transitive. Thus if a semigroup has a faithful transitive representation by relations, then every nonzero ideal of this semigroup has such a representation. A converse to this statement holds too.

Lemma 4. *If an ideal of a semigroup has a faithful transitive representation by relations, then the semigroup itself has a faithful transitive representation.*

Proof. Let Q be a faithful transitive representation of an ideal I of a semigroup $\mathcal{S}$ by relations on a set A. Without loss of generality, assume that $Q(s) \neq \emptyset$ for every $s \in I$. We can do that because if $Q(s) = \emptyset$ for some $s \in I$, then, since $\emptyset$ is the zero of $\mathcal{B}_A$ and Q is faithful, s is the zero of $\mathcal{S}$. In such a case replace I by $\{0\}$, where $0 = s$, and Q by a faithful transitive representation Q_0 of $\{0\}$ by relations on the set $A_0 = \{0\}$, where $Q_0(0) = \{(0,0)\}$.

Clearly, a labeled multi-graph $\mu : S \to \mathcal{B}_A$, defined by $\mu(s) = Q(s)$ if $s \in I$ and $\mu(s) = \emptyset$ if $s \notin I$, is a transitive quasi-representation of $\mathcal{S}$. Fix $u \in I$ and, for every $s \notin I$, choose $(a_s, b_s) \in Q(usu)$. This is possible because $Q(usu) \neq \emptyset$. Now add two new vertices i_s and j_s to A and three new labeled arrows $a_s \xrightarrow{u} i_s \xrightarrow{s} j_s \xrightarrow{u} b_s$ to μ. Let $\hat{A}$ denote the new extended set of vertices and $\hat{\mu}$ the new labeled multi-graph with the set of vertices $\hat{A}$ and old and new labeled arrows. There is a labeled path in $\hat{\mu}$ between any two vertices of $\hat{A}$, so $Q_{\hat{\mu}}$ is a transitive quasi-representation.

Now we prove that $Q_{\hat{\mu}}(r) \cap (A \times A)$ is $Q(r)$ for $r \in I$ and $\emptyset$ for $r \notin I$. Indeed, if $(i,j) \in Q_{\hat{\mu}}(r) \cap (A \times A)$, then there is a path π in $\hat{\mu}$ of length r from i to j. If all the vertices of π belong to A, then π is a path in Q, and so $(i,j) \in Q(r)$ and $r \in I$. If π contains new vertices adjoined to A, then, since the endpoints of π belong to A, these new vertices appear only in subpaths of the form $a_s \xrightarrow{u} i_s \xrightarrow{s} j_s \xrightarrow{u} b_s$ occurring in π. Replace each of these subpaths by a single labeled arrow $a_s \xrightarrow{usu} b_s$ of the same length. Such arrows belong to Q, and in this way we replace π by a new path π' in Q with $l(\pi) = l(\pi') = r$. Therefore, $(i, j) \in Q(r)$ and $r \in I$. In particular, $Q_{\hat{\mu}}(r) \cap (A \times A) = \emptyset$ for $r \notin I$. It remains to notice that $Q(r) \subset Q_{\hat{\mu}}(r) \cap (A \times A)$ for all $r \in I$.

To see that $Q_{\hat{\mu}}$ is injective, let $Q_{\hat{\mu}}(s) = Q_{\hat{\mu}}(t)$. Then $Q_{\hat{\mu}}(s) \cap (A \times A) = Q_{\hat{\mu}}(t) \cap (A \times A)$. Thus either $s, t \in I$, or $s, t \notin I$. In the former case, $Q(s) = Q(t)$, and hence $s = t$ because Q is faithful. In the latter case, $(i_s, j_s) \in Q_{\hat{\mu}}(s) = Q_{\hat{\mu}}(t)$, and there exists a labeled path π in $\hat{\mu}$ from i_s to j_s of length t. However, the only arrow beginning at i_s is $i_s \xrightarrow{s} j_s$. If $s \neq t$, then π must contain more than one labeled arrow. The only arrow beginning at j_s is $j_s \xrightarrow{u} b_s$. Therefore, π begins with $i_s \xrightarrow{s} j_s \xrightarrow{u} b_s$. Thus $l(\pi) = sux$ for some $x \in S$. Since $u \in I$, we obtain $t = l(\pi) \in I$, which is a contradiction. Thus $s = t$.

By Lemma 3, $Q_{\hat{\mu}}$ extends to a transitive representation R_ω. Also,

$$R_\omega(s) = R_\omega(t) \Rightarrow Q_{\hat{\mu}}(s) = R_\omega(s) \cap (\hat{A} \times \hat{A})$$

$$= R_\omega(t) \cap (\hat{A} \times \hat{A}) = Q_{\hat{\mu}}(t) \Rightarrow s = t,$$

and hence R_ω is a faithful transitive representation of $\mathcal{S}$ by binary relations. □

Definition. The *kernel* of a semigroup is its smallest ideal (if it exists). A semigroup $\mathcal{S}$ is called *simple* if it coincides with its kernel. Equivalently, $\mathcal{S}$ is simple when, given any $s, t \in S$, there exist $x, y \in S$ such that $t = xsy$.

Lemma 5. *Every semigroup without a kernel possesses a faithful transitive representation.*

Proof. It is a modification of the proof of Lemma 4. Let $A = \{0\}$ be a singleton set and μ a transitive representation of a semigroup $\mathcal{S}$ without a kernel by relations on A defined by $\mu(s) = \{(0, 0)\}$ for all $s \in S$. For every $t \in S$, choose an ideal I_t such that $t \notin I_t$ and choose $u_t \in I_t$.

For every pair (s, t) with $s \neq t$, add two new vertices $i_{s,t}$ and $j_{s,t}$ to A and three labeled arrows $0 \xrightarrow{u_t} i_{s,t} \xrightarrow{s} j_{s,t} \xrightarrow{u_t} 0$ to μ. Let $\hat{A}$ denote the new extended set of vertices and $\hat{\mu} : S \to \mathcal{B}_{\hat{A}}$ the new labeled multi-graph. Obviously, $Q_{\hat{\mu}}$ is transitive. Extend it to a representation R_ω freely generated by $\hat{\mu}$. By Lemmas 2 and 3, R_ω is transitive. If $R_\omega(s) = R_\omega(t)$ for some $s, t \in S$, then $Q_{\hat{\mu}}(s) = Q_{\hat{\mu}}(t)$. If $s \neq t$, then $(i_{s,t}, j_{s,t}) \in Q_{\hat{\mu}}(s) = Q_{\hat{\mu}}(t)$. Thus there exists a labeled path π in $\hat{\mu}$ from $i_{s,t}$ to $j_{s,t}$ of length t. However, the only arrow beginning at $i_{s,t}$ is $i_{s,t} \xrightarrow{s} j_{s,t}$. Also, π contains more than one labeled arrow, because $s \neq t$. The only arrow beginning at $j_{s,t}$ is $j_{s,t} \xrightarrow{u_t} 0$. Therefore, π begins with $i_{s,t} \xrightarrow{s} j_{s,t} \xrightarrow{u_t} 0$. Thus $l(\pi) = su_t x$ for some $x \in S$. Since $u_t \in I_t$, we obtain $t = l(\pi) \in I_t$, contradicting $t \notin I_t$. Thus $s = t$ and R_ω is a faithful transitive representation of S. □

To complete the proof of Theorem A, it remains to consider semigroups with kernel.

Step III. Here we define a quasi-representation $Q_{\tilde{\mu}}$ of a simple semigroup $\mathcal{S}$.

Let $E(\mathcal{S})$ be the set of all idempotents of $\mathcal{S}$ (this set may be empty). For each $e \in E(\mathcal{S})$, let G_e denote the maximal subgroup of $\mathcal{S}$ in which e is the identity element. Thus $G_e = \{s \in S \mid e \in sSs \ \wedge \ ese = s\}$.

For each maximal subgroup G_e, let $\mu_e : G_e \to \mathcal{B}_{G_e}$ be its Cayley right regular representation. The labeled arrows of μ_e are $g \xrightarrow{h} gh$ for all $g, h \in G_e$. Define $\mathcal{G}(\mathcal{S}) = \bigcup\{G_e \mid e \in E(\mathcal{S})\}$ and $\mu = \bigcup\{\mu_e \mid e \in E(\mathcal{S})\}$. The labeled arrows of

$\mu : \mathcal{G}(\mathcal{S}) \to \mathcal{B}_{\mathcal{G}(\mathcal{S})}$ are $g \xrightarrow{h} gh$ for all g and h belonging to the same subgroup of $\mathcal{S}$. Extend μ to S by defining $\mu(s) = \emptyset$ for all $s \notin \mathcal{G}(\mathcal{S})$.

Fix an idempotent $e_o \in E(\mathcal{S})$. We shall add new labeled arrows to μ connecting G_{e_o} with G_e in both directions, thus turning μ into a multi-graph $\tilde{\mu}$ such that $Q_{\tilde{\mu}}$ is injective on subgroups of $\mathcal{S}$.

For every $e \in E(\mathcal{S})$ there exist $r, s \in S$ such that $e_o = res$. Let $a_e = e_ore$ and $b_e = ese_o$. Also, assume that $a_{e_o} = b_{e_o} = e_o$. Then $e_o = a_eb_e$, $e_oa_e = a_ee = a_e$, and $eb_e = b_ee_o = b_e$ for every $e \in E(\mathcal{S})$. Also, $(b_ea_e)^2 = b_ea_eb_ea_e = b_ee_oa_e = b_ea_e$, and hence $b_ea_e \in E(\mathcal{S})$. Let $e' = b_ea_e$. Recall that $e \leq f \Leftrightarrow e = ef = fe$ is an order relation on $E(\mathcal{S})$. Obviously, $e' \leq e$ for all $e \in E(\mathcal{S})$. In particular, $e_o' = b_oa_o = e_o$.

Define $q_{e,f} = b_ea_f$ for $e, f \in E(\mathcal{S})$. Thus $e' = q_{e,e}$.

For every $e \neq e_o$ add to μ two new labeled arrows $e_o \xrightarrow{a_e} e$ and $e \xrightarrow{b_e} e_o$. Let $\tilde{\mu} : S \to \mathcal{B}_{\mathcal{G}(\mathcal{S})}$ denote the extended labeled multi-graph. There are labeled paths in $\tilde{\mu}$ from any vertex to e_o and from e_o to any vertex. Combining them, we obtain paths from any vertex to any vertex. Therefore, the quasi-representation $Q_{\tilde{\mu}}$ of $\mathcal{S}$ is transitive.

Before we proceed, we prove Lemma 9, which gives exact lengths of paths in $\tilde{\mu}$. First we prove Lemmas 6–8, which are special cases of Lemma 9 and which give lengths of certain cycles, that is, paths with coinciding endpoints. Lemma 9 is stronger than what we need for completing the proof of Theorem A, but this lemma is used later in the proof of Theorem B.

Lemma 6. *If $\pi :\ e \to \cdots \to e$ is a labeled cycle in $\tilde{\mu}$ with all vertices belonging to G_e, then all labels of π belong to G_e and $l(\pi) = e$.*

Proof. If $i \xrightarrow{a} j$ is a labeled arrow in $\tilde{\mu}$ such that $i, j \in G_e$, then $a \in G_e$. Thus π is a path in μ_e. Since μ_e is the Cayley representation of G_e, we obtain $e\big(l(\pi)\big) = e$, and hence $l(\pi) = e$. □

Lemma 7. *If $\pi : e_o \to \cdots \to e_o$ and $\tau :\ e \to \cdots \to e$, $e_o \neq e \in E(\mathcal{S})$, are labeled cycles in $\tilde{\mu}$ with vertices in $G_{e_o} \cup G_e$ such that some vertices of τ actually belong to G_{e_o}, then $l(\pi) = e_o$ and $l(\tau) = e'$.*

Proof. Obviously, π is a concatenation of one or more cycles π_k with endpoints e_o that contain no other occurrences of e_o. It suffices to show that $l(\pi_k) = e_o$ for every k. If all the vertices of π_k belong to G_{e_o}, then $l(\pi_k) = e_o$ by Lemma 6. Suppose that π_k has vertices not belonging to G_{e_o}. The only labeled arrow leading from G_{e_o} to G_e is $e_o \xrightarrow{a_e} e$. Having passed it, we make a (possibly trivial) cycle σ in G_e returning to e, and then pass the arrow $e \xrightarrow{b_e} e_o$. By Lemma 6, $l(\pi_k) = a_el(\sigma)b_e = a_eeb_e = a_eb_e = e_o$.

Also, τ is a concatenation of a cycle τ_1 in G_e, an arrow $e \xrightarrow{b_e} e_o$, a cycle $\tau_2 : e_o \to \cdots \to e_o$, an arrow $e_o \xrightarrow{a_e} e$, and a cycle τ_3 in G_e. By Lemma 6, $l(\tau_1) = e = l(\tau_3)$. By the first part of Lemma 7, $l(\tau_2) = e_o$. Thus $l(\pi) = eb_ee_oa_ee = ee'e = e'$. □

Lemma 8. *Let $\pi :\ e \to \cdots \to e$ be a labeled cycle in $\tilde{\mu}$ with vertices in $\mathcal{G}(\mathcal{S})$. If all vertices of π are in G_e, then $l(\pi) = e$. Otherwise, $l(\pi) = e'$.*

Proof. If all vertices of π are in G_e, apply Lemma 6. Let π have vertices not belonging to G_e. If $e = e_o$, then π is a concatenation of cycles π_k with endpoints e_o, where each of π_k has no other occurrences of e_o. If e_o, e_1, and e_2 are three different idempotents, then any path from e_1 to e_2 passes through e_o. Therefore, either π_k

is a path with all vertices in G_{e_o}, or it is a path with vertices in $G_{e_o} \cup G_e$ for some $e \in E(\mathcal{S})$. By Lemma 7, $l(\pi_k) = e_o$. Thus $l(\pi) = l(\pi_1)l(\pi_2)\ldots = e_oe_o\ldots = e_o$.

Now let $e \neq e_o$. If π has no occurrences of e, except the endpoints, then it is of the form $e \xrightarrow{b_e} e_o \to \cdots \to e_o \xrightarrow{a_e} e$. As we have just seen, the length of $e_o \to \cdots \to e_o$ is e_o, and hence $l(\pi) = b_ee_oa_e = b_ea_e = e'$. In general, π is a concatenation of cycles π_k with endpoints e, where π_k contain no other occurrences of e. Then $l(\pi_k)$ is e or e', and $l(\pi)$ is a product of idempotents e and e'. Since $ee' = e'e = e'$, we obtain $l(\pi) = e'$. □

Lemma 9. *Let π be a labeled path from i to j in $\tilde{\mu}$, where $i \in G_e$ and $j \in G_f$ for certain idempotents e and f of a simple semigroup $\mathcal{S}$. If all vertices of π belong to G_e, then $l(\pi) = i^{-1}j$. Otherwise, $l(\pi) = i^{-1}q_{e,f}j$.*

Proof. If all the vertices of π are in G_e, then π is a path in μ_e, and hence $i\big(l(\pi)\big) = j$. Thus $l(\pi) \in G_e$ and $l(\pi) = i^{-1}j$. If some vertices of π are not in G_e, then π is a concatenation of paths π_1 from i to e, σ from e to f, and π_2 from f to j, where all the vertices of π_1 are in G_e and all those of π_2 in G_f. Since π_1 and π_2 are paths in μ_e and μ_f, respectively, we obtain $l(\pi_1) = i^{-1}e = i^{-1}$ and $l(\pi_2) = e^{-1}j = j$. It remains to find $l(\sigma)$.

If $e = f$, then, by Lemma 8, $l(\sigma) \in \{e, e'\}$. Suppose that $e \neq f$. We cannot get from e to f without passing e_o. Therefore, σ is a concatenation of a loop σ_1 from e to e with all of its vertices in G_e, the labeled arrow $e \xrightarrow{b_e} e_o$ (if $e \neq e_o$), a loop σ_2 from e_o to e_o, the labeled arrow $e_o \xrightarrow{a_f} f$ (if $f \neq e_o$), and a loop σ_3 from f to f with all of its vertices in G_f. By Lemma 6, $l(\sigma_1) = e$ and $l(\sigma_3) = f$. By Lemma 8, $l(\sigma_2) = e_o$. Therefore, $l(\sigma) = eb_ee_oa_ff = b_ea_f = q_{e,f}$. It follows that $l(\pi) = l(\pi_1)l(\sigma)l(\pi_2) = i^{-1}q_{e,f}j$. □

Define $Q_e(s) = Q_{\tilde{\mu}}(s) \cap G_e \times G_e$ for all $s \in G_e$.

Lemma 10. *$Q_e = \mu_e$ for all $e \in E(\mathcal{S})$.*

Proof. Obviously, $\mu_e(s) \subset \mu(s) \subset \tilde{\mu}(s) \subset Q_{\tilde{\mu}}(s)$, and so $\mu_e(s) \subset Q_e(s)$. If $(i,j) \in Q_e(s)$ for $i,j,s \in G_e$, then there exists a labeled path $\pi : i \to \cdots \to j$ in $\tilde{\mu}$ of length s. This path is a concatenation of a path π_1 from i to e with vertices in G_e, followed by a cycle π_2 from e to e with vertices in $\mathcal{G}(\mathcal{S})$, and a path π_3 from e to j with vertices in G_e. By Lemma 8, $l(\pi_2) \in \{e, e'\}$.

Suppose that $l(\pi_2) = e'$. Let $l(\pi_1) = u$ and $l(\pi_3) = v$, where $u, v \in G_e$. Then $ue'v = l(\pi_1)l(\pi_2)l(\pi_3) = l(\pi_1\pi_2\pi_3) = l(\pi) = s \in G_e$. If $x \in G_e$, let x^{-1} be the inverse of x in G_e. Then $e' = ee'e = u^{-1}ue'vv^{-1} = u^{-1}sv^{-1} \in G_e$, and hence $e' = e$, because G_e has only one idempotent.

It follows that $l(\pi_1\pi_3) = l(\pi_1)l(\pi_3) = l(\pi_1)e(\pi_3) = l(\pi_1)l(\pi_2)l(\pi_3) = l(\pi_1\pi_2\pi_3) = l(\pi) = s$, where $\pi_1\pi_3$ is a path from i to j with all of its vertices in G_e. Thus $(i,j) \in \mu_e(s)$, and hence $Q_e(s) \subset \mu_e(s)$. □

Lemma 11. *Every simple semigroup $\mathcal{S}$ possesses a transitive quasi-representation Q such that the restriction $Q|_{G_e}$ of Q to each G_e is injective.*

Proof. If $\mathcal{G}(\mathcal{S}) = \emptyset$, let $Q(s) = \{(0,0)\}$ for all $s \in S$. Obviously, Q is a transitive representation of $\mathcal{S}$ by relations on a set $\{0\}$. Also, Q is injective on all subgroups of $\mathcal{S}$ (because there are none). If $\mathcal{G}(\mathcal{S}) \neq \emptyset$, use Step III to construct the quasi-representation $Q = Q_{\tilde{\mu}}$. We have seen that Q is transitive. If $Q(s) = Q(t)$ for some $s, t \in G_e$ and $e \in E(\mathcal{S})$, then, by Lemma 10, $\mu_e(s) = Q_e(s) = Q(s) \cap G_e \times G_e =$

$Q(t) \cap G_e \times G_e = Q_e(t) = \mu_e(t)$, and hence $s = t$, because the Cayley representation μ_e is injective. Thus Q is injective on maximal subgroups of $\mathcal{S}$. □

Step IV. Here we define a quasi-representation $Q_{\check{\nu}}$.

Let $\nu : S \to \mathcal{B}_A$ be a transitive quasi-representation of a simple semigroup $\mathcal{S}$ such that ν is injective on all subgroups of $\mathcal{S}$ and $A \neq \emptyset$, and let $X = \{(s,t) \in S \times S | \, \nu(s) = \nu(t) \wedge s \neq t \wedge s \notin tsSst\}$.

If $s \in tsSst$ and $t \in stSts$ for some $s, t \in S$, then $s = tsxst$ and $t = styts$ for suitable $x, y \in S$. Let $a = sxst$, $b = tyts$, $c = tsxs$, and $d = styt$. Then $s = ta = ct$ and $t = sb = ds$, so that $t = tab$ and $s = ct = ctab = sab$. Analogously, $cds = s$, and hence $cdt = t$. Therefore, $ab = sx \cdots = (cds)x \cdots = (cd)(ab) = (\cdots yt)(ab) = \cdots y(tab) = \cdots yt = cd$. The element $e = ab = cd$ is idempotent, and hence $s, t \in G_e$. If $\nu(s) = \nu(t)$, then $s = t$, because ν is injective on G_e. Thus, if $\nu(s) = \nu(t)$ and $s \neq t$, then $(s,t) \in X$ or $(t,s) \in X$.

Since $A \neq \emptyset$ and ν is transitive, we can choose $w \in S$ and $i, j \in A$ such that $(i,j) \in \nu(w)$. Next choose $u_{s,t}, v_{s,t} \in S$ such that $w = u_{s,t}stsv_{s,t}$.

For every pair $(s,t) \in X$ add four new vertices $a_{s,t}$, $b_{s,t}$, $c_{s,t}$ and $d_{s,t}$ to A. Let $\check{A}$ be the extended set of vertices. Also, for every $(s,t) \in X$ add to ν the following five labeled arrows: $i \xrightarrow{u_{s,t}} a_{s,t} \xrightarrow{s} b_{s,t} \xrightarrow{t} c_{s,t} \xrightarrow{s} d_{s,t} \xrightarrow{v_{s,t}} j$. Let $\check{\nu}$ denote the new labeled multi-graph $S \to \mathcal{B}_{\check{A}}$ and let $Q_{\check{\nu}}$ be the quasi-representation of $\mathcal{S}$ corresponding to the multigraph $\check{\nu}$. Clearly, $\check{\nu}$ is transitive, and hence $Q_{\check{\nu}}$ is transitive.

Lemma 12. *Every simple semigroup admits a faithful transitive representation by binary relations.*

Proof. Suppose that $Q_{\check{\nu}(s)} = Q_{\check{\nu}(t)}$ for some $s, t \in S$. If s and t belong to the same subgroup of $\mathcal{S}$, then $s = t$ because $\check{\nu}$ is injective on the subgroups. Suppose that $s \neq t$ and s and t do not belong to the same subgroup. Then $(s,t) \in X$ or $(t,s) \in X$. Without loss of generality, let $(s,t) \in X$. Then $(b_{s,t}, c_{s,t}) \in \check{\nu}(t) \subset Q_{\check{\nu}(t)}$, and hence $(b_{s,t}, c_{s,t}) \in Q_{\check{\nu}(s)}$. Thus there exists a labeled path π of length s in $\check{\nu}$ leading from $b_{s,t}$ to $c_{s,t}$. Since $s \neq t$, this path contains more than one arrow. However, the only path in $\check{\nu}$ leading from $b_{s,t}$ that has more than one arrow is $b_{s,t} \xrightarrow{t} c_{s,t} \xrightarrow{s} d_{s,t} \to \cdots$, and the only path leading to $c_{s,t}$ that is not a single arrow is $\cdots \to a_{s,t} \xrightarrow{s} b_{s,t} \xrightarrow{t} c_{s,t}$. Therefore, $s = l(\pi) = ts...st \in tsSst$, which contradicts $(s,t) \in X$. Thus $s = t$.

Let R_ω be a free representation of $\mathcal{S}$ generated by $\check{\nu}$. It is transitive and faithful because it extends the transitive and injective $Q_{\check{\nu}}$. □

Now we can complete the proof of Theorem A. If a semigroup $\mathcal{S}$ has no kernel, apply Lemma 5. If $\mathcal{S}$ has a kernel K, then K is a simple semigroup. Apply Lemma 12 and then Lemma 4.

A remark-essay on "metrics" in multi-graphs

To prove Theorem A we used a "metric" on multi-graphs. This idea can be (and has been) applied to other situations, this is why we discuss it now in more detail. As we have already seen, if we want to consider "metrics" in multi-graphs, it is natural to assume that the length of a path should be the "sum" of lengths of its arrows in the order in which they are passed. Thus it is natural to assume that the set S, from which we take "lengths" of arrows, has a binary operation (following tradition, we call the result of its application a "product" rather than "sum"). Looking at the length of a three-arrow path $\cdot \xrightarrow{x} \cdot \xrightarrow{y} \cdot \xrightarrow{z} \cdot$, we see that it is natural to assume that $(xy)z = x(yz)$ for all $x, y, z \in S$. Thus it is natural to use *semigroups* (or at least small categories) as sets of possible distances in multi-graphs.

If we want our "metrics" to satisfy other "natural" properties, we may have to impose specific conditions on the semigroup $\mathcal{S}$ of "distances." For example, it may be natural to ask what is the length of a trivial path with a single vertex and no arrows in it. Our "metric intuition" tells us this is "zero." Analyzing that intuition, we see that the length in this case should not "add anything" to the length of any path. Thus, if the length of an "empty" path (i) with a single vertex i is a and π is a path of length s beginning at i, then the length of the concatenation of (i) and π should be the same as $l(\pi)$, that is, $as = s$. Concatenating (i) with itself, we see that $aa = a$, that is, a is an idempotent. Analogously, considering paths of length t ending at i and concatenating them with (i), we see that $ta = t$. Thus, a is an idempotent left identity or a right identity for certain elements of $\mathcal{S}$. If we prefer the length of (i) not to depend on the choice of the vertex i, than this length must be the identity element of $\mathcal{S}$; that is, it is natural to assume that $\mathcal{S}$ is a monoid. If the length of (i) depends on i, we obtain a semigroup with specific conditions (for example, each of its elements has left and right idempotent identities).

What about the "symmetry" of our "metric"? One obstacle is that the arrows in our multi-graphs are oriented and passed in the direction of their orientation in any path. It is possible to ask what might be the length of an arrow $i \xrightarrow{s} j$ if it is passed in the opposite direction, from j to i. One possibility is to consider non-oriented arrows. Then, if we want our metric to be symmetric, we have to conclude that the length st of a path $i \overset{s}{-} j \overset{t}{-} k$ should coincide with the length of that path passed in the opposite direction, from k to i, and hence $st = ts$. Thus we have to assume that our "metric semigroup" $\mathcal{S}$ is commutative, and we may use a more intuitively appealing additive terminology for lengths of paths.

However, in certain situations (for example, for representations by binary relations) arrows of our multi-graphs are oriented and commutativity of $\mathcal{S}$ may not be a natural condition. Another approach is possible. Assume that if two arrows $i_1 \longrightarrow j_1$ and $i_2 \longrightarrow j_2$ have the same length s, then those arrows passed in the opposite direction also have the same length, say s^{-1}. In other words, assume that the length of an arrow $i \xrightarrow{s} j$ passed in the opposite direction depends on s but not on the endpoints i and j. Then $s \to s^{-1}$ is a unary operation in $\mathcal{S}$. If we reverse the orientation of an arrow $\xrightarrow{s}$ twice, its length s will change as follows: $s \to s^{-1} \to (s^{-1})^{-1}$. However, after two reversals we return to the original arrow, and hence it is natural to assume that $(s^{-1})^{-1} = s$ for any $s \in S$.

If $\pi : \cdot \xrightarrow{s} \cdot \xrightarrow{t} \cdot$ is a path of length st, it is natural to assume that its length in the opposite direction would be $(st)^{-1}$. However, it is no less natural to assume that the length of π in the opposite direction would be the product $t^{-1}s^{-1}$. Thus we obtain $(st)^{-1} = t^{-1}s^{-1}$ for any $s, t \in S$.

An algebra of the form $\mathcal{S} = (S; \cdot, ^{-1})$ is called an *involuted semigroup* if $(S; \cdot)$ is a semigroup, $^{-1}$ is a unary operation, and the identities $(x^{-1})^{-1} = x$ and $(xy)^{-1} = y^{-1}x^{-1}$ hold in $\mathcal{S}$. Thus, when we measure lengths of arrows in the direction opposite to their orientation, we may consider involuted semigroups.

If the distances in a multi-graph are measured by elements of an involuted semigroup, we may determine lengths of all sorts of "zigzags." For example, consider the configuration $i \xrightarrow{s} j \xleftarrow{t} k \xrightarrow{u} l$. This is not a path, but we can turn it into a path by reversing the orientation of $j \longleftarrow k$. The length of that path would be $st^{-1}u$.

Now suppose that we want to pass an arrow $i \xrightarrow{s} j$ from i to j, then return to i, and then return back to j. The length of this "zigzag path" would be $ss^{-1}s$. Is this zigzag path "longer" than the original arrow $i \xrightarrow{s} j$? To compare lengths of different paths we need something like an order relation, we need a binary relation $s \prec t$, that means that s is "shorter than or equal to" t. Obviously, this relation should be reflexive: $s \prec s$, and transitive: $s \prec t \wedge t \prec u \Rightarrow s \prec u$. It may be antisymmetric: if s is shorter than t and t shorter than s, then $s = t$, but situations are conceivable in which $\prec$ does not have to be antisymmetric. Also, $\prec$ does not have to be linear: neither $s \prec t$ nor $t \prec s$ may hold for certain $s, t \in S$. So, if we want to compare distances, our semigroup $\mathcal{S}$ should be equipped with a (partial) order relation, or at least with a quasi-order relation. There may be other natural conditions too. For example, if $i \xrightarrow{s_1} j$ is shorter than $i \xrightarrow{s_2} j$ and $j \xrightarrow{t_1} k$ is shorter than $j \xrightarrow{t_2} k$, then it is natural to assume that $i \xrightarrow{s_1} j \xrightarrow{t_1} k$ is shorter than $i \xrightarrow{s_2} j \xrightarrow{t_2} k$. In other words, $s_1 \prec s_2 \wedge t_1 \prec t_2 \Rightarrow s_1t_1 \prec s_2t_2$, that is, $\prec$ is *stable* (or *compatible with multiplication*): $(\prec)(\prec) \subset (\prec)$, where $\prec$ is considered as a subset of the semigroup $\mathcal{S} \times \mathcal{S}$. What can we say about s^{-1} and t^{-1} if $s \prec t$? We should not jump to the conclusion $t^{-1} \prec s^{-1}$, typical for a group situation, because $^{-1}$ does not necessarily denote the "reciprocal" or "inverse." Indeed, if $i \xrightarrow{s} j$ is shorter than $i \xrightarrow{t} j$, we may conclude that $i \xleftarrow{s^{-1}} j$ is shorter than $i \xleftarrow{t^{-1}} j$, that is, $s \prec t \Rightarrow s^{-1} \prec t^{-1}$.

Also, we can interpret $l(\pi)$ for a path π as the "work" done when we move along π. In particular, we may consider "conservative fields" in which lengths of cycles are "zero," or at least an idempotent element. For example, if the work s is done when we move along an arrow $i \longrightarrow j$, then $i \xrightarrow{s} j$ is the corresponding labeled arrow, and we can conclude that the work is "undone" when we move back from j to i along the same arrow. Thus, the length ss^{-1} of the path $i \xrightarrow{s} j \longleftarrow i$ is e, where e is the identity element of $\mathcal{S}$ (or at least e is a "local identity"). This approach leads us to groups and inverse semigroups.

On the other hand, geometric intuition may tell us that $s \prec ss^{-1}s$ for all $s \in S$ because the length of a "zigzag" $i \xrightarrow{s} j \xleftarrow{s} i \xrightarrow{s} j$ should be longer than or equal to the length of $i \xrightarrow{s} j$. This is a particular case of the "triangle inequality," and we may look into what the triangle inequality means in our situation. This is a fruitful approach, for it led to a satisfactory solution of another longstanding semigroup-theoretic problem. An involuted semigroup $\mathcal{S}$ is called *representable by binary relations* if it is isomorphic to an involuted semigroup $(\Phi; \circ, ^{-1})$ of binary relations on a set. Here, if $\varphi \in \Phi$, then $\varphi^{-1} = \{(i, j) \mid (j, i) \in \varphi\}$. The problem of characterizing involuted semigroups representable by binary relations, first raised in 1953, was solved in [6] using an aproach quite analogous to that used in the proof of Theorem A. It turns out that the representability of an involuted semigroup $\mathcal{S}$ is equivalent to a certain "triangular inequality" property for the metric on labeled multi-graphs associated with $\mathcal{S}$. This property gives rise to a system of quasi-identities that characterize axiomatically the class of representable involuted semigroups.

Another application of certain ideas used in the proof of Theorem A yielded a proof of the following theorem (see [1]): for every (partially) ordered semigroup

$\mathcal{S} = (S; \cdot; \leq)$ there exists an isomorphism P onto an inclusion-ordered semigroup of binary relations $(\Phi; \circ; \subset)$ such that, for any nonempty subset $T \subset S$ for which the greatest lower bound $a = \inf T$ exists in $\mathcal{S}$, $\mathrm{P}(a) = \bigcap\{\mathrm{P}(t) \mid t \in T\}$ (thus P is an infima-preserving isomorphism). In particular, the construction used in [1] leads to a representation P such that all the relations $\mathrm{P}(s)$ satisfy certain special conditions. It would be interesting to see if the construction from [1] could yield transitive representations by special types of relations.

Undoubtedly, metrics in labeled graphs can be used for solving other problems concerning semigroups of binary relations.

Proof of Theorem B

Recall that the set of idempotents of a semigroup is ordered by $\leq$ (defined in Step III). A *completely simple semigroup* is a simple semigroup with a minimal idempotent. Historically, finite simple semigroups (which are obviously completely simple) were the first class of semigroups, besides groups, for which a nontrivial structural theorem was found (see [7] and [8], or Appendix A in [2]).

In Step III we saw that $e' \leq e$ for idempotents e of a simple semigroup. In the completely simple case all idempotents are primitive (see [7], [8], or [3]), and hence $e' = e$. Also, if $e \in E(\mathcal{S})$ and $es \in G_e$ for an element s of a completely simple semigroup $\mathcal{S}$, then $s \in G_e$ (for example, see Theorem 2.52(iii) of [2].) In this case Lemma 9 can be restated as follows.

Lemma 13. *If $s \in S$, $i \in G_e$, and $j \in G_f$ for some $e, f \in E(\mathcal{S})$ in a completely simple semigroup $\mathcal{S}$, then the following are equivalent:*
(i) $(i, j) \in Q_{\tilde{\mu}}(s)$;
(ii) $s = i^{-1}q_{e,f}j$;
(iii) $is = q_{e,f}j$;
(iv) $j = q_{f,e}is$.

Proof. The alternative lengths in Lemma 9 coincide because $e' = e$ for all $e \in E(\mathcal{S})$. By Lemma 9, *(i)* $\Leftrightarrow$ *(ii)*. If *(ii)* holds, then $is = ii^{-1}q_{e,f}j = eq_{e,f}j = q_{e,f}j$, which is *(iii)*. Also, $q_{e,f}q_{f,k} = b_ea_fb_fa_k = b_ee_oa_k = b_ea_k = q_{e,k}$ for any $e, f, k \in E(\mathcal{S})$. In particular, $q_{f,e}q_{e,f} = q_{f,f} = f$. Thus *(iii)* implies $q_{f,e}is = q_{f,e}q_{e,f}j = fj = j$, which is *(iv)*. If *(iv)* is true, then $s = es = i^{-1}eis = i^{-1}q_{e,f}q_{f,e}is = i^{-1}q_{e,f}j$, which is *(ii)*. □

Lemma 14. *For a completely simple semigroup, $Q_{\tilde{\mu}}$ is a faithful transitive representation by binary relations.*

Proof. A completely simple semigroup is a union of its maximal subgroups. Let H_s denote the maximal subgroup that contains $s \in S$ (thus $H_e = G_e$ for $e \in E(\mathcal{S})$). Then $H_sH_t \subset H_{st}$ and $H_sH_tH_s \subset H_s$ for all $s, t \in S$ (see [2]).

Let $(i, j) \in Q_{\tilde{\mu}}(st)$ for some $s, t \in S$, $i \in G_e$, and $j \in G_f$. By Lemma 13, $j = q_{f,e}ist$. Let g be the identity element of the subgroup H_{ts}. Define $k = q_{g,e}is = gq_{g,e}is \in H_{ts}SH_s \subset H_tH_sSH_s \subset H_{ts} = G_g$. By Lemma 13, $(i, k) \in Q_{\tilde{\mu}}(s)$. Also, $kt = q_{g,e}ist = q_{g,f}q_{f,e}ist = q_{g,f}j$. By Lemma 13, $(k, j) \in Q_{\tilde{\mu}}(t)$. It follows that $(i, j) \in Q_{\tilde{\mu}}(s) \circ Q_{\tilde{\mu}}(t)$, so that $Q_{\tilde{\mu}}(st) \subset Q_{\tilde{\mu}}(s) \circ Q_{\tilde{\mu}}(t)$, and hence $Q_{\tilde{\mu}}(st) = Q_{\tilde{\mu}}(s) \circ Q_{\tilde{\mu}}(t)$ for all $s, t \in S$, because $Q_{\tilde{\mu}}$ is a transitive quasi-representation. Thus $Q_{\tilde{\mu}}$ is a transitive representation.

Suppose that $Q_{\tilde{\mu}}(s) = Q_{\tilde{\mu}}(t)$ for some $s, t \in S$. If $s \in G_e$, then $s = es = eq_{e,e}s$, and, by Lemma 13, $(e, s) \in Q_{\tilde{\mu}}(s)$. Therefore, $(e, s) \in Q_{\tilde{\mu}}(t)$, and hence $t = eq_{e,e}s = s$. Therefore, $Q_{\tilde{\mu}}$ is faithful. □

Remark. The representation $Q_{\tilde{\mu}}$ for completely simple semigroups was defined (with a different notation) in [5]. We mention only a few of its remarkable properties. A representation $R : S \to \mathcal{B}_A$ is called *simply transitive* if, for any $(a, b) \in A \times A$, there exists *exactly one* s in $\mathcal{S}$ such that $(a, b) \in R(s)$. As proved in [5], a semigroup $\mathcal{S}$ has a faithful simply transitive representation if and only if it is completely simple or completely 0-simple (the latter case occurs when $\mathcal{S}$ has a zero represented by the empty binary relation). Thus completely simple semigroups are *characterized* by the fact of being isomorphic to simply transitive semigroups of nonempty binary relations. If $\mathrm{P}\colon \mathcal{S} \to \mathcal{B}_A$ is a faithful simply transitive representation of a completely simple semigroup, *inflate* it as follows. For a set B and a mapping $\varphi : B \to A$ of B *onto* A, define $\mathrm{P}_\varphi : S \to \mathcal{B}_B$ by the formula $(b, c) \in \mathrm{P}_\varphi(s) \Leftrightarrow (\varphi(b), \varphi(c)) \in \mathrm{P}(s)$ for all $b, c \in B$. Thus every pair $(i, j) \in A \times A$ is replaced by a "rectangle" $\varphi^{-1}(i) \times \varphi^{-1}(j) \subset B \times B$. It is not difficult to see that P_φ is a faithful simply transitive representation of $\mathcal{S}$. It turns out that *every* faithful simply transitive representation of $\mathcal{S}$ is an inflation of a representation (called *canonical*) that is not a nontrivial inflation of any representation. Any two canonical representations of $\mathcal{S}$ are similar. Here representations $R_1 : S \to \mathcal{B}_{A_1}$ and $R_2 : S \to \mathcal{B}_{A_2}$ are called *similar* if there exists a bijection $\theta : A_1 \to A_2$ such that $(a, b) \in R_1(s) \Leftrightarrow (\theta(a), \theta(b)) \in R_2(s)$ for all $s \in S$ and $a, b \in A_1$. It is easy to see that the representation $Q_{\tilde{\mu}}$ from Lemma 14 is simply transitive. It turns out that it is exactly the canonical representation of $\mathcal{S}$ introduced in [5].

Let $\mathcal{S}$ be a semigroup with a completely simple kernel K. Extend the canonical representation $Q_{\tilde{\mu}}$ of K to a mapping $R : S \to \mathcal{B}_K$ by defining $(i, j) \in R(s) \Leftrightarrow is = q_{e,f}j$ for every $s \in S$, $i \in G_e$, and $j \in G_f$, where $e, f \in E(K)$.

Lemma 15. *R is a transitive representation of $\mathcal{S}$. It is injective on K, and $R(s) \neq \emptyset$ for all $s \in S$.*

Proof. Let $(i, j) \in R(s)$ and $(j, k) \in R(t)$ for $i \in G_e$, $j \in G_f$, and $k \in G_g$, where G_e, G_f, and G_g are certain maximal subgroups of K. Then $is = q_{e,f}j$ and $jt = q_{f,g}k$ in K. It follows that $ist = q_{e,f}jt = q_{e,f}q_{f,g}k = q_{e,g}k$, and hence $(i, k) \in R(st)$.

Suppose that $(i, k) \in R(st)$ for some $i \in G_e$ and $k \in G_g$. Then $kg = k$ and $ist = q_{e,g}k$, whence $ist = q_{e,g}k = q_{e,g}kg = istg$. Also, $is, tg \in K$, and, by Lemma 13, $(i, k) \in Q_{\tilde{\mu}}(is \cdot tg) = Q_{\tilde{\mu}}(is) \circ Q_{\tilde{\mu}}(tg)$. Thus $(i, j) \in Q_{\tilde{\mu}}(is)$ and $(j, k) \in Q_{\tilde{\mu}}(tg)$ for some $j \in G_f$, where f is a suitable idempotent of K. By Lemma 13, $is = q_{e,f}j$ and $jtg = q_{f,g}k$. It follows from $jf = j$ that $isf = q_{e,f}jg = q_{e,f}j = is$. Principal left ideals of a completely simple semigroup are minimal (see [2]). Thus $ft \in Kft = Kis \cdot ft = Kist = Kistg \subset Ktg$, and hence $ft = xtg$ for some $x \in K$. It follows that $jt = jft = jxtg = jxtgg = jtg = q_{f,g}t$. Therefore, $(i, j) \in R(s)$ and $(j, k) \in R(t)$, which implies $(i, k) \in R(s) \circ R(t)$. Thus $R(st) = R(s) \circ R(t)$, so that R is a representation of $\mathcal{S}$. By the definition of R and Lemma 13, R coincides with $Q_{\tilde{\mu}}$ on K, and, by Lemma 14, R is injective on K. Also, R is transitive because $Q_{\tilde{\mu}}$ is transitive.

It is easy to see that $(e_o, e_o) \in Q_{\tilde{\mu}}(e_o)$. Since K is simple, it follows that $e_o = xty$ for each $t \in K$ and some $x, y \in K$. Thus $Q_{\tilde{\mu}}(e_o) = Q_{\tilde{\mu}}(x) \circ Q_{\tilde{\mu}}(t) \circ Q_{\tilde{\mu}}(y)$, and hence $Q_{\tilde{\mu}}(t) \neq \emptyset$. Finally, $ks \in K$ for every $k \in K$ and $s \in S$. It follows that $R(k) \circ R(s) = R(ks) = Q_{\tilde{\mu}}(ks) \neq \emptyset$, and hence $R(s) \neq \emptyset$. $\square$

Lemma 16. *Every finite semigroup with zero admits a faithful transitive representation by nonempty binary relations on a finite set.*

Proof. In the proof of Lemma 4 we have already constructed a faithful transitive representation for a semigroup $\mathcal{S}$ with zero 0. Here we produce another representation. Let M be the set of all subsets of $\mathcal{S}^1$ that contain 0, where $\mathcal{S}^1$ is the semigroup obtained from $\mathcal{S}$ by adjoining an identity 1 (see [2]).

Define $Z : S \to \mathcal{B}_M$ as follows: $(\alpha, \beta) \in Z(s) \Leftrightarrow \alpha s \subset \beta$ for any $\alpha, \beta \in M$ and $s \in S$. If $(\alpha, \gamma) \in Z(s) \circ Z(t)$, then $(\alpha, \beta) \in Z(s)$ and $(\beta, \gamma) \in Z(t)$ for some $\beta \in M$, so that $\alpha s \subset \beta$ and $\beta t \subset \gamma$. It follows that $\alpha st \subset \beta t \subset \gamma$, and hence $(\alpha, \gamma) \in Z(st)$. Conversely, suppose that $(\alpha, \gamma) \in Z(st)$, that is, $\alpha st \subset \gamma$. Let $\beta = \alpha s$. Then $0 = 0s \in \alpha s = \beta$, so that $\beta \in M$. Obviously, $(\alpha, \beta) \in Z(s)$ and $(\beta, \gamma) \in Z(t)$. It follows that $Z(s) \circ Z(t) = Z(st)$ for all $s, t \in S$. Clearly, $(\alpha, \beta) \in Z(0)$ for all $\alpha, \beta \in M$. Thus Z is a transitive representation of $\mathcal{S}$ by binary relations. If $s \in S$, then $\{0, 1\}s = \{0, s\}$, and so $(\{0, 1\}, \{0, s\}) \in Z(s)$. It follows that $Z(s) \neq \emptyset$.

If $Z(s) = Z(t)$ for $s, t \in S$, then $(\{0, 1\}, \{0, s\}) \in Z(t)$, so that $t \in \{0, 1\}t \subset \{0, s\}$. Interchanging the roles of s and t, we obtain $s \in \{0, t\}$. It follows that $s = t$, and hence the representation Z is faithful. It remains to observe that, if $\mathcal{S}$ is finite, then M is finite. $\square$

Remark. The representation Z considered in the proof of Lemma 16 first appeared in [4].

Now we can complete the proof of Theorem B. A finite semigroup $\mathcal{S}$ has a completely simple kernel K. By Lemma 15, R is a transitive representation of $\mathcal{S}$ by relations on K.

Consider the Rees factor semigroup $T = \mathcal{S}/K$. Its elements are $(S \backslash K) \cup \{0\}$, where 0 is the zero and the multiplication in $S \backslash K$ is defined in the same way as in $\mathcal{S}$, except that whenever the product belongs to K, it is 0 (see [2]). Then T is a finite semigroup with zero and a homomorphic image of $\mathcal{S}$ under an obvious homomorphism $\varphi : S \to T$. By Lemma 16, T has a faithful transitive representation Z by binary relations on a finite set M. Combining φ and Z, we obtain a transitive representation $\zeta = \varphi \circ Z$ of $\mathcal{S}$.

If $R_i : S \to \mathcal{B}_{A_i}$ are representations of $\mathcal{S}$ for $i = 1, 2$, construct a representation $R_1 \Box R_2 : S \to \mathcal{B}_{A_1 \times A_2}$ as follows: $\big((a_1, a_2), (b_1, b_2)\big) \in R_1 \Box R_2(s) \Leftrightarrow (a_i, b_i) \in R_i(s)$ for $i = 1, 2$, $a_i, b_i \in A_i$, and $s \in S$.

Let $\rho = R \Box \zeta$. It is a representation of $\mathcal{S}$ by binary relations on a set $K \times M$. Let $(i, \alpha), (j, \beta) \in K \times M$. There exists $s \in K$ such that $(i, j) \in Q_{\tilde{\mu}}(s) = R(s)$. Also, $\varphi(s) = 0 \in T$, and $Z(0) = M \times M$ (see our proof of Lemma 16). Thus $(\alpha, \beta) \in \zeta(s)$, and hence $\big((i, \alpha), (j, \beta)\big) \in \rho(s)$. It follows that ρ is a transitive representation of $\mathcal{S}$.

Suppose that $\rho(s) = \rho(t)$ for some $s, t \in S$; then $R(s) \times \zeta(s) = R(t) \times \zeta(t)$ and, since $R(s)$ and $\zeta(s)$ are nonempty for all $s \in S$, we obtain $R(s) = R(t)$ and $\zeta(s) = \zeta(t)$. If $s \notin K$ or $t \notin K$, then $\zeta(s) = \zeta(t)$ implies $s = t$. If $s, t \in K$, then $R(s) = R(t)$ implies $s = t$, by Lemma 15. Thus ρ is faithful.

Since $\mathcal{S}$ is finite, both K and M are finite, and hence $K \times M$ is finite. $\square$

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